

Inverse boundary value problem for Schrödinger equation in two dimensions

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Abstract

We relax the regularity condition on potentials of the Schrödinger equation in uniqueness results on the inverse boundary value problem which were recently proved in [11] and [5].

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain with $\partial\Omega = \cup_{j=0}^K \Sigma_j$ where Σ_j are smooth contours and Σ_0 is the external contour. Let $\nu = (\nu_1, \nu_2)$ be the unit outer normal to $\partial\Omega$ and let $\frac{\partial}{\partial\nu} = \nabla \cdot \nu$.

In this domain we consider the Schrödinger equation with some potential q :

$$(\Delta + q)u = 0 \quad \text{in } \Omega. \quad (1)$$

Let $\tilde{\Gamma}$ be a non-empty arbitrary fixed relatively open subset of $\partial\Omega$. Denote $\Gamma_0 = \text{Int}(\partial\Omega \setminus \tilde{\Gamma})$. Consider the partial Cauchy data

$$\mathcal{C}_q = \left\{ \left(u, \frac{\partial u}{\partial \nu} \right) \Big|_{\tilde{\Gamma}} ; (\Delta + q)u = 0 \quad \text{in } \Omega, u|_{\Gamma_0} = 0, u|_{\tilde{\Gamma}} = f \right\}. \quad (2)$$

The goal of this article is to improve the regularity assumption on the potential q in the case of arbitrary subboundary $\tilde{\Gamma}$ for the uniqueness result in the inverse problem of recovery of potential from the partial data (2). In the case of $\tilde{\Gamma} = \partial\Omega$, this inverse problem was formulated by Calderón in [7]. Under the assumption $q \in C^{4+\alpha}(\overline{\Omega})$ the result was proved in Imanuvilov, Uhlmann and Yamamoto [11]. In Guillarmou and Tzou [10], the assumption on potentials was improved up to $C^{2+\alpha}(\overline{\Omega})$.

In particular, in the two-dimensional full Cauchy data case of $\tilde{\Gamma} = \partial\Omega$, we refer to Astala and Päivärinta [1], Blasten [2], Brown and Uhlmann [4], Bukhgeim [5], Nachman [14]. In [2], the full Cauchy data uniquely determine the potential within $W_p^1(\Omega)$ with $p > 2$. As for the related problem of recovery of the conductivity, [1] proved the uniqueness result for conductivities from $L^\infty(\Omega)$, improving the result of [14]. We also mention that for the case of full Cauchy data a relaxed regularity assumption on potential was claimed in [5] but the proof itself is missing some details.

In three or higher dimensions, for the full Cauchy data, Sylvester and Uhlmann [16] proved the uniqueness of recovery of conductivity in $C^2(\overline{\Omega})$, and later the regularity assumption was relaxed up to $C^{\frac{3}{2}}(\overline{\Omega})$ in Päivärinta, Panchenko and Uhlmann [15] and up to $W_p^{\frac{3}{2}}(\Omega)$ with $p > 2n$ in Brown and Torres [3]. For the case of partial Cauchy data, uniqueness theorems were proved under assumption that a potential of the Schrödinger equation belongs to $L^\infty(\Omega)$ (see Bukhgeim and Uhlmann [6], Kenig, Sjöstrand and Uhlmann [13]).

Our main result is as follows

Theorem 1 *Let $q_1, q_2 \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$ if $\tilde{\Gamma} = \partial\Omega$ and $q_1, q_2 \in W_p^1(\Omega)$ for some $p > 2$ otherwise. If $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then $q_1 = q_2$.*

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The rest part of the paper is devoted to the proof of the theorem. Throughout the article, we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $D = (\frac{1}{i}\partial_{x_1}, \frac{1}{i}\partial_{x_2})$. The tangential derivative on the boundary is given by $\partial_{\vec{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial\Omega$.

Proof.

First Step.

Let $\Phi = \varphi + i\psi$ be a holomorphic function on Ω such that φ, ψ are real-valued and

$$\Phi \in C^2(\bar{\Omega}), \quad \text{Im } \Phi|_{\Gamma_0} = 0. \quad (3)$$

Denote by \mathcal{H} the set of the critical points of the function Φ . Suppose that this set is not empty, each critical point is nondegenerate, $\mathcal{H} \cap \bar{\Gamma}_0 = \emptyset$ and

$$\text{mes}(\mathcal{J}) = 0, \quad \mathcal{J} = \{x; \partial_{\vec{\tau}}\psi(x) = 0, x \in \tilde{\Gamma}\}. \quad (4)$$

Here $\vec{\tau}$ is an unit tangential vector to $\partial\Omega$. Consider the operator $L_q(x, D) = -\sum_{j=1}^2 (D_j + \tau i\varphi_{x_j})^2 + q$. It is known (see [12] Proposition 2.5) that there exists a constant τ_0 such that for $|\tau| \geq \tau_0$ and any $f \in L^2(\Omega)$, there exists a solution to the boundary value problem

$$L_q(x, D)u = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0 \quad (5)$$

such that

$$\|u\|_{H^{1,\tau}(\Omega)} \sqrt{|\tau|} \leq C \|f\|_{L^2(\Omega)}. \quad (6)$$

Moreover if $f/\partial_z\Phi \in L^2(\Omega)$, then for any $|\tau| \geq \tau_0$ there exists a solution to the boundary value problem (5) such that

$$\|u\|_{H^{1,\tau}(\Omega)} \leq C \|f/\partial_z\Phi\|_{L^2(\Omega)}. \quad (7)$$

The constants C in (6) and (7) are independent of τ . Here and henceforth we set

$$\|u\|_{H^{1,\tau}(\Omega)} = (\|u\|_{H^1(\Omega)}^2 + |\tau|^2 \|u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

Second Step.

Here we will construct complex geometrical optics solutions. Henceforth by $o_{L^2(\Omega)}(\frac{1}{\tau})$, we mean a function $f(\epsilon, \tau, \cdot) \in L^2(\Omega)$ such that $\lim_{\tau \rightarrow \infty} |\tau| \|f(\epsilon, \tau, \cdot)\|_{L^2(\Omega)} = 0$ for all small $\epsilon > 0$, and by $o(\frac{1}{\tau})$, we mean $a(\epsilon, \tau)$ such that $\lim_{\tau \rightarrow \infty} |\tau| |a(\epsilon, \tau)| = 0$ for all small $\epsilon > 0$.

Let $\{q_{1,\epsilon}\}_{\epsilon \in (0,1)}$ be a sequence of smooth functions converging to q_1 in $W_p^1(\Omega)$ or $C^\alpha(\bar{\Omega})$ (depending on the assumption on the regularity of q_1) such that $q_{1,\epsilon} = q_1$ on \mathcal{H} . Let p_ϵ be the complex geometrical optics solution to the Schrödinger operator $\Delta + q_{1,\epsilon}$ which we constructed in [11]. The function p_ϵ can be written in the form:

$$p_\epsilon(x) = e^{\tau\Phi}(a + a_{0,\epsilon}/\tau) + e^{\tau\bar{\Phi}}\overline{(a + b_{1,\epsilon}/\tau)} - \left(e^{\tau\Phi} \frac{(\partial_{\bar{z}}^{-1}(aq_{1,\epsilon}) - M_{1,\epsilon})}{4\tau\partial_z\Phi} + e^{\tau\bar{\Phi}} \frac{(\partial_z^{-1}(\bar{a}q_{1,\epsilon}) - M_{3,\epsilon})}{4\tau\bar{\partial}_z\Phi} \right) + e^{\tau\varphi} o_{L^2(\Omega)}\left(\frac{1}{\tau}\right) \quad \text{as } \tau \rightarrow +\infty, \quad (8)$$

where $a \in C^6(\bar{\Omega})$ is some holomorphic function on Ω such that $\text{Re } a|_{\Gamma_0} = 0$. The operators ∂_z^{-1} and $\partial_{\bar{z}}^{-1}$ are given by

$$\partial_{\bar{z}}^{-1}g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\zeta, \bar{\zeta})}{\zeta - z} d\xi_2 d\xi_1, \quad \partial_z^{-1}g = \overline{\partial_{\bar{z}}^{-1}\bar{g}},$$

Moreover for some $\tilde{x} \in \mathcal{H}$, we assume that $a(\tilde{x}) \neq 0$ and $a(x) = 0$ for $x \in \mathcal{H} \setminus \{\tilde{x}\}$, and the polynomials $M_{1,\epsilon}(z)$ and $M_{3,\epsilon}(\bar{z})$ satisfy

$$\partial_z^j(\partial_{\bar{z}}^{-1}(aq_{1,\epsilon}) - M_{1,\epsilon})(x) = 0, \quad \partial_{\bar{z}}^j(\partial_z^{-1}(\bar{a}q_{1,\epsilon}) - M_{3,\epsilon})(x) = 0, \quad x \in \mathcal{H},$$

$a_{0,\epsilon}, a_{1,\epsilon} \in C^6(\overline{\Omega})$ are holomorphic functions such that

$$(a_{0,\epsilon} + \bar{a}_{1,\epsilon})|_{\Gamma_0} = \frac{(\partial_z^{-1}(aq_{1,\epsilon}) - M_{1,\epsilon})}{4\partial_z\Phi} + \frac{(\partial_z^{-1}(\bar{a}q_{1,\epsilon}) - M_{3,\epsilon})}{4\bar{\partial}_z\bar{\Phi}}.$$

We look for a solution u_1 in the form $u_1 = p_\epsilon + m_\epsilon$. Consider the equation

$$L_{q_1}(x, D)u_1 = L_{q_{1,\epsilon}}(x, D)(p_\epsilon + m_\epsilon) + (q_1 - q_{1,\epsilon})(p_\epsilon + m_\epsilon) = L_{q_1}(x, D)m_\epsilon + (q_1 - q_{1,\epsilon})p_\epsilon = 0.$$

By (7) there exists a solution to the boundary value problem

$$L_{q_1}(x, D)m_\epsilon + (q_1 - q_{1,\epsilon})p_\epsilon = 0 \quad \text{in } \Omega, \quad m_\epsilon|_{\Gamma_0} = 0$$

such that

$$\|m_\epsilon\|_{H^{1,\tau}(\Omega)} \leq C(\epsilon) \quad \forall \tau > \tau_0(\epsilon), \quad (9)$$

where $C(\epsilon)$ is independent of τ and

$$C(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Since the Cauchy data (2) for potentials q_1 and q_2 , are equal, there exists a solution u_2 to the Schrödinger equation with the potential q_2 such that $u_1 = u_2$ on $\partial\Omega$ and $\frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu}$ on $\tilde{\Gamma}$. Setting $u = u_1 - u_2$, we obtain

$$(\Delta + q_2)u = (q_2 - q_1)u_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}|_{\tilde{\Gamma}} = 0. \quad (10)$$

In a way similar to the construction of u_1 , we construct the complex geometrical optics solution v for the Schrödinger equation with the potential q_2 . The construction of v repeats the corresponding steps of the construction of u_1 . The only difference is that instead of $q_{1,\epsilon}$ and τ , we use $q_{2,\epsilon}$ and $-\tau$ respectively. We provide details of the construction of v for the sake of completeness.

Let $\{q_{2,\epsilon}\}_{\epsilon \in (0,1)}$ be a sequence of smooth functions converging to sufficiently close to q_2 in $W_p^1(\Omega)$ or $C^\alpha(\overline{\Omega})$ such that $q_{2,\epsilon} = q_2$ on \mathcal{H} . Let \tilde{p}_ϵ be the complex geometrical optics solution to the Schrödinger operator $\Delta + q_{2,\epsilon}$ constructed in [11]:

$$\begin{aligned} \tilde{p}_\epsilon(x) &= e^{-\tau\Phi}(a + b_{0,\epsilon}/\tau) + e^{-\tau\bar{\Phi}}\overline{(a + b_{1,\epsilon}/\tau)} \\ &+ \left(e^{-\tau\Phi} \frac{(\partial_z^{-1}(aq_{2,\epsilon}) - M_{2,\epsilon})}{4\tau\partial_z\Phi} + e^{-\tau\bar{\Phi}} \frac{(\partial_z^{-1}(\bar{a}q_{2,\epsilon}) - M_{4,\epsilon})}{4\tau\bar{\partial}_z\bar{\Phi}} \right) + e^{-\tau\varphi} o_{L^2(\Omega)}\left(\frac{1}{\tau}\right), \end{aligned} \quad (11)$$

where $M_{2,\epsilon}(z)$ and $M_{4,\epsilon}(\bar{z})$ satisfy

$$\partial_z^j(\partial_z^{-1}(aq_{1,\epsilon}) - M_{2,\epsilon})(x) = 0, \quad \partial_z^j(\partial_z^{-1}(\bar{a}q_{1,\epsilon}) - M_{4,\epsilon})(x) = 0, \quad x \in \mathcal{H}.$$

and $b_{0,\epsilon}, b_{1,\epsilon}$ are holomorphic functions such that

$$(b_{0,\epsilon} + \bar{b}_{1,\epsilon})|_{\Gamma_0} = -\frac{(\partial_z^{-1}(aq_{2,\epsilon}) - M_{2,\epsilon})}{4\partial_z\Phi} - \frac{(\partial_z^{-1}(\bar{a}q_{2,\epsilon}) - M_{4,\epsilon})}{4\bar{\partial}_z\bar{\Phi}}.$$

We look for a solution v in the form $v = \tilde{p}_\epsilon + \tilde{m}_\epsilon$. Consider the operator

$$L_{q_2}(x, D)v = L_{q_{2,\epsilon}}(x, D)(\tilde{p}_\epsilon + \tilde{m}_\epsilon) + (q_2 - q_{2,\epsilon})(\tilde{p}_\epsilon + \tilde{m}_\epsilon) = L_{q_2}(x, D)\tilde{m}_\epsilon + (q_2 - q_{2,\epsilon})\tilde{p}_\epsilon = 0.$$

By (7) there exists a solution to the boundary value problem

$$L_{q_2}(x, D)\tilde{m}_\epsilon + (q_2 - q_{2,\epsilon})\tilde{p}_\epsilon = 0 \quad \text{in } \Omega, \quad \tilde{m}_\epsilon|_{\Gamma_0} = 0$$

such that

$$\|\tilde{m}_\epsilon\|_{H^{1,\tau}(\Omega)} \leq C(\epsilon) \quad \forall \tau > \tau_0(\epsilon), \quad (12)$$

where $C(\epsilon)$ is independent of τ and

$$C(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Third Step.

We will prove $q_1(\tilde{x}) = q_2(\tilde{x})$ where $a(\tilde{x}) \neq 0$ and $a(x) = 0$ for $x \in \mathcal{H} \setminus \{\tilde{x}\}$ in the case where $q_1, q_2 \in W_p^1(\Omega)$.

Denote $q = q_1 - q_2$. Taking the scalar product of equation (10) and the function v , we have:

$$\int_{\Omega} qu_1 v dx = 0. \quad (13)$$

By (9) and (12)

$$0 = \int_{\Omega} qu_1 v dx = \int_{\Omega} qp_{\epsilon} \tilde{p}_{\epsilon} dx + K(\epsilon, \tau), \quad (14)$$

where

$$\overline{\lim_{\tau \rightarrow +\infty}} \tau |K(\epsilon, \tau)| \leq C(\epsilon), \quad C(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (15)$$

From (14), (15) and the explicit formulae (8), (11) for the construction of complex geometrical optics solutions, we have

$$\int_{\Omega} q(a^2 + \bar{a}^2) dx = 0.$$

Computing the remaining terms, we have:

$$\begin{aligned} K(\epsilon, \tau) + \frac{1}{\tau} \int_{\Omega} q(a(a_{0,\epsilon} + b_{0,\epsilon}) + \overline{a(a_{1,\epsilon} + b_{1,\epsilon})}) dx + \int_{\Omega} q(a\bar{a}e^{2\tau i\psi} + a\bar{a}e^{-2\tau i\psi}) dx \\ + \frac{1}{4\tau} \int_{\Omega} \left(qa \frac{\partial_z^{-1}(aq_{2,\epsilon}) - M_{2,\epsilon}}{\partial_z \Phi} + q\bar{a} \frac{\partial_z^{-1}(q_{2,\epsilon}\bar{a}) - M_{4,\epsilon}}{\partial_z \bar{\Phi}} \right) dx \\ - \frac{1}{4\tau} \int_{\Omega} \left(qa \frac{\partial_z^{-1}(q_{1,\epsilon}a) - M_{1,\epsilon}}{\partial_z \Phi} + q\bar{a} \frac{\partial_z^{-1}(q_{1,\epsilon}\bar{a}) - M_{3,\epsilon}}{\partial_z \bar{\Phi}} \right) dx \\ + o\left(\frac{1}{\tau}\right) = 0 \quad \text{as } \tau \rightarrow +\infty. \end{aligned} \quad (16)$$

Since the functions q_j are not supposed to be from $C^2(\bar{\Omega})$, we can not directly use the stationary phase argument (e.g., Evans [8]). Consider two cases. Assume that $q \in W_p^1(\Omega)$ with $p > 2$. We have

$$\int_{\Omega} q \text{Re}(a\bar{a}e^{2\tau i\psi}) dx = \int_{\Omega} q_{\epsilon} \text{Re}(a\bar{a}e^{2\tau i\psi}) dx + \int_{\Omega} (q - q_{\epsilon}) \text{Re}(a\bar{a}e^{2\tau i\psi}) dx. \quad (17)$$

We set $q_{\epsilon} = q_{1,\epsilon} - q_{2,\epsilon}$. Taking into account that $q_{j,\epsilon} = q_j$ on \mathcal{H} , $j = 1, 2$, (4) and using the stationary phase argument, similar to [11], we compute

$$\int_{\Omega} q_{\epsilon}(a\bar{a}e^{2\tau i\psi} + a\bar{a}e^{-2\tau i\psi}) dx = \frac{2\pi(q|a|^2)(\tilde{x}) \text{Re } e^{2\tau i \text{Im } \Phi(\tilde{x})}}{\tau |(\det \text{Im } \Phi'')(\tilde{x})|^{\frac{1}{2}}} + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \rightarrow +\infty. \quad (18)$$

For the second integral in (17) we obtain

$$\begin{aligned} \int_{\Omega} (q - q_{\epsilon})(a\bar{a}e^{2\tau i\psi} + a\bar{a}e^{-2\tau i\psi}) dx &= \int_{\Omega} (q - q_{\epsilon}) \left(a\bar{a} \frac{(\nabla \psi, \nabla) e^{2\tau i\psi}}{2\tau i |\nabla \psi|^2} - a\bar{a} \frac{(\nabla \psi, \nabla) e^{-2\tau i\psi}}{2\tau i |\nabla \psi|^2} \right) dx \\ &= \int_{\partial \Omega} (q - q_{\epsilon}) \left(a\bar{a} \frac{(\nabla \psi, \nu) e^{2\tau i\psi}}{2\tau i |\nabla \psi|^2} - a\bar{a} \frac{(\nabla \psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla \psi|^2} \right) d\sigma \\ &\quad - \frac{1}{2\tau i} \int_{\Omega} \left\{ e^{2\tau i\psi} \text{div} \left((q - q_{\epsilon}) a\bar{a} \frac{\nabla \psi}{|\nabla \psi|^2} \right) - e^{-2\tau i\psi} \text{div} \left((q - q_{\epsilon}) a\bar{a} \frac{\nabla \psi}{|\nabla \psi|^2} \right) \right\} dx. \end{aligned} \quad (19)$$

Since $\psi|_{\Gamma_0} = 0$ we have

$$\int_{\partial \Omega} (q - q_{\epsilon}) a\bar{a} \left(\frac{(\nabla \psi, \nu) e^{2\tau i\psi}}{2\tau i |\nabla \psi|^2} - \frac{(\nabla \psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla \psi|^2} \right) d\sigma = \int_{\tilde{\Gamma}} \frac{(q - q_{\epsilon}) a\bar{a}}{2\tau i |\nabla \psi|^2} (\nabla \psi, \nu) (e^{2\tau i\psi} - e^{-2\tau i\psi}) d\sigma.$$

By (4) and Proposition 2.4 in [11] we have that

$$\int_{\partial\Omega} (q - q_\epsilon) a \bar{a} \left(\frac{(\nabla\psi, \nu) e^{2\tau i\psi}}{2\tau i |\nabla\psi|^2} - \frac{(\nabla\psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla\psi|^2} \right) d\sigma = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \rightarrow +\infty.$$

The last integral over Ω in formula (19) is $o(\frac{1}{\tau})$ and so

$$\int_{\Omega} (q - q_\epsilon) (a \bar{a} e^{2\tau i\psi} + a \bar{a} e^{-2\tau i\psi}) dx = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \rightarrow +\infty. \quad (20)$$

Taking into account that $\psi(\tilde{x}) \neq 0$ and using (26), (20) we have from (16) that

$$\frac{2\pi(q|a|^2)(\tilde{x})}{|(\det \operatorname{Im} \Phi'')(\tilde{x})|^{\frac{1}{2}}} + \tilde{C}(\epsilon) = 0, \quad (21)$$

where $\tilde{C}(\epsilon) \rightarrow +0$ as $\epsilon \rightarrow 0$. Hence

$$q(\tilde{x}) = 0 \quad \text{if } a(\tilde{x}) \neq 0 \text{ and } a(x) = 0 \text{ for } x \in \mathcal{H} \setminus \{\tilde{x}\}. \quad (22)$$

Since a point \tilde{x} can be chosen arbitrarily close to any given point in Ω (see [11]), we have $q \equiv 0$, that is, the proof of the theorem is completed if $q_1, q_2 \in W_p^1(\Omega)$.

Fourth Step.

Now let $q \in C^\alpha(\bar{\Omega})$ with some $\alpha \in (0, 1)$ and $\partial\Omega = \tilde{\Gamma}$.

We recall the following classical result of Hörmander [9]. Consider the "oscillatory integral operator"

$$T_\tau f(x) = \int_{\Omega} e^{-\tau i\psi(x,y)} a(x,y) f(y) dy,$$

where $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and $a(\cdot, \cdot) \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. We introduce the following matrix

$$H_\psi = \{\partial_{x_i y_j}^2 \psi\}.$$

Theorem 2 *Suppose that $\det H_\psi \neq 0$ on $\operatorname{supp} a$. Then*

$$\|T_\tau\|_{L^2 \rightarrow L^2} \leq \frac{C}{\tau}.$$

Consider our holomorphic function $\Phi(x, y) = (x_1 + ix_2 - (y_1 + iy_2))^2 + i$. We set $\psi(x, y) = 2(x_1 - y_1)(x_2 - y_2) - 1$. Then

$$H_\psi(x, y) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

and $\det H_\psi(x, y) = -4$. Then the condition in Theorem 2 holds true.

We set $a(x, y) = \chi(x)\chi(y)$ where $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\chi|_\Omega \equiv 1$. Then, by Theorem 2, there exists a constant C independent of τ such that

$$\|T_\tau\|_{L^2 \rightarrow L^2} + \|T_{-\tau}\|_{L^2 \rightarrow L^2} \leq C/\tau. \quad (23)$$

Setting $f = (q - q_\epsilon) a \bar{a} \chi_\Omega$ by (23) we have

$$\|T_\tau f\|_{L^2(\Omega)} + \|T_{-\tau} f\|_{L^2(\Omega)} \leq C(\epsilon)/\tau, \quad C(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow +0. \quad (24)$$

Therefore, by (24), in the ball $B(\tilde{x}, \delta) \equiv \{x; |x - \tilde{x}| < \delta\}$, there exists a sequence of points $y(\tau)$ such that

$$|(T_\tau)f(y(\tau))| + |(T_{-\tau})f(y(\tau))| \leq \frac{C\epsilon}{\tau\delta^2}. \quad (25)$$

Let $y(\tau) = (y_1(\tau), y_2(\tau)) \rightarrow \hat{y}(\epsilon)$ as $\tau \rightarrow +\infty$. By the stationary phase argument taking into account that $\psi(\tilde{x}, \tilde{x}) = -1$, we have

$$\int_{\Omega} (q_{\epsilon} - (q_{\epsilon} - q)(y(\tau)) \operatorname{Re}\{a\bar{a}\epsilon^{-2\tau i\psi(y(\tau), x)}\}) dx = \frac{2\pi(q|a|^2)(\hat{y}(\epsilon)) \operatorname{Re} e^{2\tau i}}{\tau} + o\left(\frac{1}{\tau}\right). \quad (26)$$

From (16), (26), (25) we obtain

$$2\pi(q|a|^2)(\hat{y}(\epsilon)) \operatorname{Re} e^{2\tau i} + \tilde{C}(\epsilon) = 0, \quad (27)$$

where $\overline{\lim}_{\tau \rightarrow +\infty} |\tilde{C}(\epsilon)| \rightarrow +0$ as $\epsilon \rightarrow 0$. Therefore as ϵ goes to zero, we have

$$q(\hat{x}) = 0.$$

Here $\hat{x} \in B(\tilde{x}, \delta)$ such that $\hat{y}(\epsilon) \rightarrow \hat{x}$ as $\epsilon \rightarrow +0$. Since $\delta > 0$ and \tilde{x} are chosen arbitrarily, we conclude that $q \equiv 0$ in Ω . Thus the proof of the theorem is completed. \square

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